$$\begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & a \\ 4 & a & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$，由此可得 $a = -1, \lambda_1 = 2$．故 $A = \begin{pmatrix} 0 & -1 & 4 \\ -1 & 3 & -1 \\ 4 & -1 & 0 \end{pmatrix}$．

由 $|\lambda E - A| = \begin{vmatrix} \lambda & 1 & -4 \\ 1 & \lambda - 3 & 1 \\ -4 & 1 & \lambda \end{vmatrix} = (\lambda + 4)(\lambda - 2)(\lambda - 5) = 0$，

可得 A 的特征值为 $\lambda_1 = 2, \lambda_2 = -4, \lambda_3 = 5$．

由 $(\lambda_2 E - A)x = 0$，即 $\begin{pmatrix} -4 & 1 & -4 \\ 1 & -7 & 1 \\ -4 & 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$，可解得对应于 $\lambda_2 = -4$ 的线性无关的特征向量为 $\xi_2 = (-1,0,1)^T$．

由 $(\lambda_3 E - A)x = 0$，即 $\begin{pmatrix} 5 & 1 & -4 \\ 1 & 2 & 1 \\ -4 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$，可解得对应于 $\lambda_3 = 5$ 的特征向量为 $\xi_3 = (1,-1,1)^T$．

由于 A 为实对称矩阵，ξ_1, ξ_2, ξ_3 为对应于不同特征值的特征向量，所以 ξ_1, ξ_2, ξ_3 相互正交，只需单位化：

$$\eta_1 = \frac{\xi_1}{\|\xi_1\|} = \frac{1}{\sqrt{6}}(1,2,1)^T, \eta_2 = \frac{\xi_2}{\|\xi_2\|} = \frac{1}{\sqrt{2}}(-1,0,1)^T, \eta_3 = \frac{\xi_3}{\|\xi_3\|} = \frac{1}{\sqrt{3}}(1,-1,1)^T,$$

取 $Q = (\eta_1, \eta_2, \eta_3) = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$，则 $Q^T A Q = \Lambda = \begin{pmatrix} 2 & & \\ & -4 & \\ & & 5 \end{pmatrix}$．